THE

MATHEMATICAL GAZETTE.

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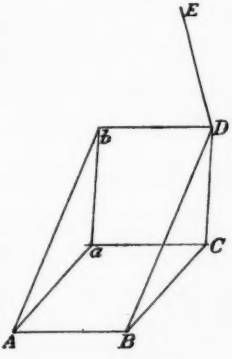
THE POLYGONS OF PONCELET AND WEILL'S THEOREM.

THIS paper contains what I believe is, at least in form, a new proof of the theorems of Poncelet and Weill relating to polygons inscribed in one circle and circumscribed to another. It depends on the following lemma:

If A is a fixed point, and $ABCDE\ldots$ is a variable broken line, whose parts AB, BC, CD, ... are all of the same constant length, and are so drawn that the lines AC, BD, CE, ..., joining each vertex to the next but one, are the same in magnitude, direction, and sense as different lines from a fixed point to a fixed circle, the same for all, then the locus of each vertex of the broken line is a circle.



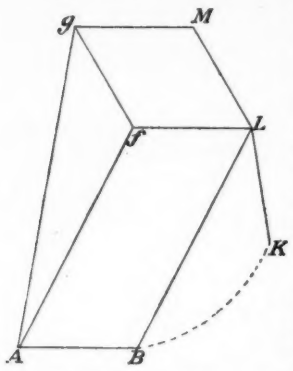
This is clearly true of B and C. To prove it for D, complete the parallelograms $ABCa$, $ABDb$. Then $aCDb$ is a parallelogram. Ab is equal and parallel to BD, and therefore the locus of b is a circle, the same as for C. Thus $abDC$ is a rhombus whose opposite vertices b, C are constrained to move on a fixed circle. Hence, as in Peaucellier's cell, the loci of a and D are inverse to each other.* But Aa is a constant length, therefore the locus of D is a circle.

*This is proved as follows: Da bisects Cb at right angles, and therefore passes through a fixed point T, viz. the centre of the fixed circle on which C and b lie. But since a, D lie on a circle whose centre is C and radius CD,

$$Ta \cdot TD = TC^2 - CD^2 = \text{constant}.$$

Hence a and D are inverse with respect to T.

In general, let K, L, M be three consecutive vertices. Complete the parallelograms $ABLf$, $ABMg$. Then $LMgf$ is a rhombus. Also g is a position of L, and f the corresponding position of K. This is seen by supposing the broken line $B \ldots KLM$ moved without deformation or rotation, so that B may come to A, thus giving a new polygon which begins at A and satisfies all the conditions required in the lemma.

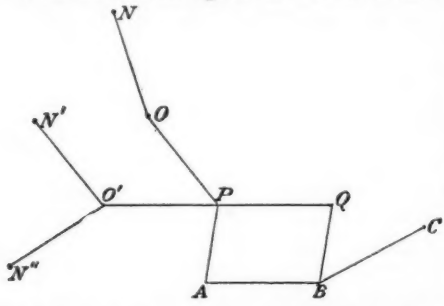


If then we suppose it proved that the locus of K (or f) and that of L (or g) are circles, it follows that the locus of M is also a circle, since $fgML$ is a rhombus. Hence it may be proved of each vertex in succession that its locus is a circle.

The centres of these circles are easily seen to be in one straight line. For the circles which K (or f) and M describe are inverse with respect to the centre of the circle L (Note, p. 121).

Suppose now that PQ, the n^{th} piece of the broken line, is parallel to AB, and in the same sense.

Let the whole figure be moved without rotation or deformation, so that B may come to A. Then Q will come to P. Suppose P, O to come to O', N' respectively, O being the vertex before P. Draw $O'N''$ equal and parallel to CB, and in the same sense.



Then N' is a position of N, O' the corresponding one of O, and P that of P. But in the induction above it was shown that N', P were inverse to each other with respect to a fixed circle. Similarly the point N'', which is obtained from $N'O'P$ in the same way as N' from NOP, is the inverse of P with respect to the same circle. Thus P is the inverse of both N' and N'', and the circle of inversion must therefore have its centre at P and radius zero. Therefore the n^{th} vertex will always fall at P.

Thus also Q, the n^{th} vertex starting from B, will be such that BQ is fixed in magnitude and direction, the same as AP, so that

$APQB$ is a parallelogram. The n^{th} side PQ will therefore always be parallel to the first AB.

Now let lines be drawn from a fixed point O the same in magnitude, direction, and sense as AB, BC, CD, ... Let these lines be $O\alpha$, $O\beta$, $O\gamma$, ..., so that the points α, β, γ, ... are ranged on a circle S_1. Let α', β', γ', ... be the middle points of $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, ...; then these points are also ranged on a circle S_2, since $O\alpha'$, $O\beta'$, ... are parallel to and equal to the halves of AC, BD, ... respectively. Hence also the intersections of tangents at α and β, β and γ, γ and δ, ... lie on a fixed circle S_3, which is the inverse of S_2 with respect to S_1.

If now PQ is parallel to AB, the n^{th} point of the series $\alpha\beta\gamma$... will coincide with α, so that the tangents will form a polygon of $(n-1)$ sides inscribed in S_3 and circumscribed to S_1. If this happens for any one position of α, it will happen for every position.

Since each point of the series B, C, D, E, ... traces a circle, it follows that the mean centre of the first r points α, β, γ, ... traces a circle of $\frac{1}{r}^{\text{th}}$ the radius; and since P is a fixed point, the mean centre of the $\overline{n-1}$ points $\alpha\beta\gamma$... is also fixed.* For instance, the line AD is parallel to and treble of the line joining O to the mean centre of $\alpha\beta\gamma$, and A, O are fixed points.

The locus of the mean centre of r consecutive vertices of the series α, β, γ, ... is a circle, but not, as sometimes stated, that of any r vertices. I find the degree of this locus to be $2s$, where s is the number of gaps made in the ring by leaving out the other $n-r-1$ vertices; there is an s-ple point at each of the focoids. For instance, the middle point of $\alpha\gamma$ traces a bicircular quartic.

A. C. DIXON.

NOTES ON ELEMENTARY DYNAMICS. IV.

THE PARALLELOGRAM OF VELOCITIES, AND RELATIVE VELOCITY.

THE aim of this note is to point out a certain looseness of method and haziness of conception which seems to me to exist in the treatment of the above subjects in our current text-books, and to suggest improvements.

In a deservedly popular text-book of the Elements of Dynamics, we find the following accurate, and nearly† complete enunciation of the "Parallelogram of Velocities":

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*This is Weill's Theorem. Compare Casey's *Sequel*, p. 164.

†It could be made quite complete by substituting "drawn from the point" for the last four words, which as they stand do not indicate the *sense*

If a moving point possess simultaneously velocities which are represented in magnitude and direction by the two sides of a parallelogram drawn from a point, they are equivalent to a velocity which is represented in magnitude and direction by the diagonal of the parallelogram passing through the point.

But the inquisitive learner will ask: What is meant by a body possessing simultaneously *two* velocities?

If we recur to the definition of velocity we shall find that it gives us no help here. We may get some light as to what is really meant, by considering the *proof* usually given, or by observing the use made of the proposition in its applications.

The usual proof involves the idea of a body *moving relatively to another*, and in fact the meaning tacitly attached to the phrase under consideration is, that the body has a motion relative to a moving base, and that the actual motion of the body is said to consist of two simultaneous velocities, viz. that of the base and that of the body relative to the base.

But as the "Parallelogram of Velocities" is usually given before Relative Motion has been discussed, and is employed in the discussion of Relative Motion, the learner's mind is often puzzled, so that he fails to grasp the force of the theorem, and its proof.

Next consider the *applications* of the "Parallelogram of Velocities." The most important of these is perhaps that in the proof of the "Parallelogram of Forces." That theorem depends on the principle that the accelerations produced by two forces acting simultaneously on the same body are *independent, i.e.* that they co-exist without affecting one another.

But here again we must ask: What is meant by two accelerations *co-existing*?

I do not see that we can avoid taking co-existing accelerations in the same sense as we took co-existing velocities, that is, considering one acceleration as that of a moving base, and the other that of a point or body *relatively* to the base—unless indeed we cut the knot by turning the "Parallelogram of Velocities" or accelerations into a *definition* of what is meant by component velocities or accelerations—in which case of course a *proof* of the "Parallelogram of Velocities" would be illogical.

I am not at all certain that the latter method of treatment

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in which the diagonal is to be taken. In this connection I should like to suggest that the type of enunciation of such theorems in which a lettered diagram (actual or imaginary) is referred to, is preferable for clearness and brevity combined: *e.g. If a moving point possess simultaneously two velocities represented in magnitude and direction by OA, OB, two lines drawn from a point O, then the resultant velocity is represented in direction and magnitude on the same scale by the diagonal OC of the completed parallelogram OACB.* The late Professor Casey adopted this type of enunciation in his *Euclid.*

would not be the best in an elementary book, if the changes necessarily accompanying it were made. It would then cease to be illogical to use the "Parallelogram of Velocities" in the investigation of Relative Velocity. But for the present I shall take the former alternative, which involves the discussion of Relative Velocity first. It might perhaps be argued that *all* motion being *relative*, it is unphilosophical to make any distinction: but in an elementary course we must I think be content to start with the idea of an immovable base, such as we instinctively regard the Earth to be. At anyrate the object of this note is not to revolutionize Elementary Dynamics, but if possible to clarify to some extent the conceptions which are actually current.

What knowledge of Relative Velocity is necessary in order to prove the "Parallelogram of Velocities"? Since we cannot strictly speak of the motion of a point relatively to a *point* (unless a base of reference for directions is supposed in addition), it would seem that we must first consider Displacements of Bodies. For simplicity let us confine ourselves to two dimensions, a plane body moving in a plane.

The following proposition is required, and easily proved:

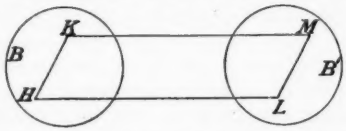
When a plane body A is displaced in its plane so that a line in it remains parallel to a certain fixed line, then every line in A remains parallel to a fixed line, and the displacement of every point in A is the same in magnitude and direction.

Such a displacement of A is called a *translation* or a *displacement without rotation.*

A similar proposition holds good for the displacement of a body A *relative* to any body B, which may or may not be in motion; only instead of *fixed line* we must say *line* fixed in B, and instead of *displacement* we must say *relative displacement.*

We can now prove the "Parallelogram of Displacements without Rotation" or "of Steps" (Clifford).

Suppose a point P to move from a point H to a point K in a body B. Suppose then the body moves without rotation to B' so that HK comes to LM. On the whole the point P has been displaced from H to M. But it is obvious that the same result is attained whether the two motions are simultaneous or not. Hence at once we get the "Parallelogram of Steps."



Note that the theorem is true whatever be the intermediate motion of B or P during the displacement, only the initial and final positions being had in view. For this reason it might be well, following Clerk Maxwell in *Matter and Motion*, to refrain at first from joining the points in displacement-diagrams.

The "Parallelogram of Velocities" may now be deduced. Let

P move relatively to B with a uniform velocity v, so that its displacement after t seconds is vt, and let B move translationally with a uniform velocity u, so that its displacement after t seconds is ut. In the same diagram as before let B' represent the position of B after t seconds. Then we have

$$\frac{ML}{HL}=\frac{vt}{ut}=\frac{v}{u}=\text{constant.}$$

Thus the angle MHL is constant, and $\dfrac{HM}{HL}$ is constant (Euc. VI. 6).

$$\therefore\ MH \propto t.$$

Thus the resultant motion of P is uniform and rectilinear, and is represented by HM in direction and magnitude on the same scale on which the velocity of B is represented by HL.

So far we have referred only to *uniform rectilinear* velocities. The proof as it stands does not apply to those which are not uniform and rectilinear. This remark applies to all proofs of this kind, such as are usually given in the text-books, and yet we find the difficulty ignored or slurred over, the result being tacitly assumed to hold for varying velocity.

To extend the proposition to varying velocities we might proceed thus: We note that a point can only then be said to have a definite velocity when by taking a small enough interval of time we find the motion up to that instant as nearly as we please uniform and rectilinear. Now when this is the case, the construction given for uniform velocities will fit our case as nearly as we please to have it, if we make the interval of time small enough. We may for clearness suppose the diagram correspondingly magnified in scale, so that it always remains finite.

A more formal proof will be found in Clifford's *Dynamic*, pp. 59, 60.

Some may object to what precedes as hypercritical. The reply is that what is given in elementary text-books ought at least to have a definite and unmistakeable *meaning*. I think it is quite an open question how far young students should be troubled with formal proofs of fundamental propositions, as such proofs must of necessity involve really difficult abstract ideas. I think clear explanations and illustrations (the illustrations preferably coming first) are at anyrate more useful. But the proofs that are given ought at least to be real proofs so far as they go, and the terms employed ought to be clearly defined where any doubt can exist. I think there is room for improvement in this respect in nearly all the current text-books.

In a further note I propose to discuss two or three problems which have been set of late years, involving relative motion.

R. F. MUIRHEAD.

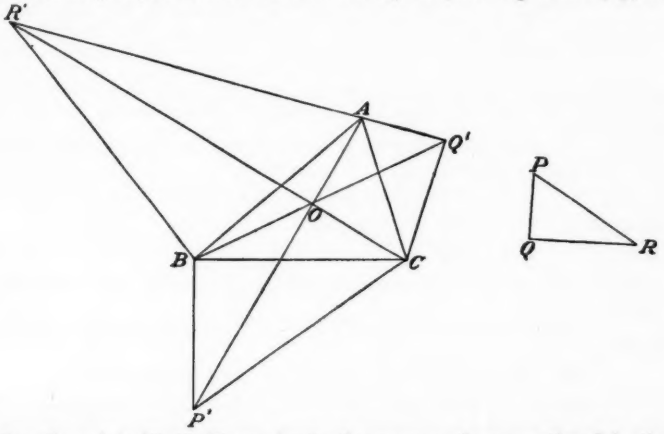
MATHEMATICAL NOTES.

44. *On a property of two triangles and its extension to quadrilaterals.*

In the following note, when we say that two triangles are similar, such as for example the two triangles $BP'C$, QPR, we mean not only that the angles B, P', C are equal to Q, P, R respectively, but also that the rotations denoted by $BP'C$, QPR are in the same sense both positive, or both negative. The following simple theorem is no doubt well known :

If ABC, PQR be any two triangles, and on the sides of ABC three other triangles $BP'C$, $CQ'A$, $AR'B$ be described similar to QPR, RQP, PRQ respectively, then the pairs of lines AP', QR, and BQ', RP, and CR', PQ have one and the same product, and are equally inclined to one and the same line.

To prove this we may take a point O within the triangle ABC such that the angles BOC, COA, AOB are the supplements of P, Q, R respectively, and produce AO, BO, CO to meet the circles BOC, COA, AOB again in P', Q', R'.



Then the points P', Q', R' are clearly the same as those mentioned in the enunciation. The angle between AP' and BQ' is the angle $P'OQ'$ which is equal in magnitude and opposite in sense to the angle between QR and RP. Hence the pairs of lines AP', QR, and BQ', RP, and similarly the pair CR', PQ, are equally inclined to one and the same line.

Also, since $OBP'C$ is a cyclic quadrilateral,

$$OP' \cdot BC = OB \cdot CP' + OC \cdot BP' ;$$

and since BC, CP', BP' are proportional to QR, RP, PQ,

therefore $$OP' \cdot QR = OB \cdot RP + OC \cdot PQ.$$

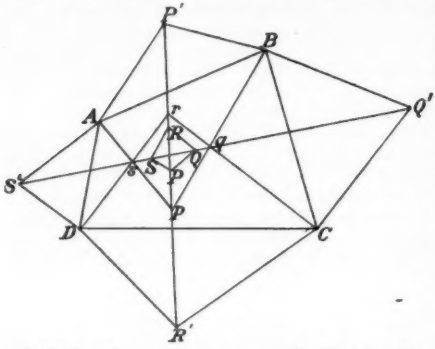
Adding $OA \cdot QR$ to each side, we have

$$AP' \cdot QR = OA \cdot QR + OB \cdot RP + OC \cdot PQ$$
$$= BQ' \cdot RP = CR' \cdot PQ.$$

This establishes the theorem for two triangles. To extend it to quadrilaterals we require to prove the following :

If $ABCD$, $PQRS$ be any two quadrilaterals, and triangles $AP'B$, $BQ'C$, $CR'D$, $DS'A$ be described similar to QPS, RQP, SRQ, PSR respectively, then the pairs of lines $P'R'$, SQ, and $Q'S'$, PR have the same product, and are equally inclined to the same line.

Suppose that only two of the triangles, viz., $AP'B$ and $CR'D$, are described in the first instance. Let $P'R'$ cut the circle $P'AB$ in p, and the circle $R'CD$ in r; let Bp and Cr meet in q, and Ap and Dr in s; also let qs meet the circle qBC in Q', and the circle sDA in S'. Then it can be easily proved that the points Q', S' are the same as those mentioned in the enunciation. For, if O is the point of intersection of pr and qs, and P, Q, R, S inverses of p, q, r, s with respect to O, it easily follows that $PQRS$ is similar to and in the same sense as the given quadrilateral $PQRS$, and that it may be taken to be the given quadrilateral for the purpose of proving the theorem. Hence it follows at once that the pairs of lines $P'R'$, SQ, and $Q'S'$, PR are equally inclined to the same line.



To prove that the products $P'R' . SQ$ and $Q'S' . PR$ are equal, we have, since $pAP'B$ is a cyclic quadrilateral,

$$P'p . AB = pA . P'B + pB . P'A ;$$

therefore
$$P'p . SQ = pA . PS + pB . PQ ;$$

similarly
$$rR' . SQ = rC . RQ + rD . RS.$$

Adding the two last equations, and subtracting $pr . SQ$ from each side, we have

$$P'R' . SQ = pA . PS + pB . PQ + rC . RQ + rD . RS - pr . SQ.$$

Similarly

$$Q'S' . PR = sA . PS + qB . PQ + qC . RQ + sD . RS + qs . RP ;$$

$$\therefore \; P'R' . SQ - Q'S' . PR = ps . PS + pq . PQ + rq . RQ + rs . RS - pr . SQ - qs . RP.$$

This last result may be shown to vanish from the fact that p, q, r, s are the inverses of P, Q, R, S with respect to the point O where pr, qs intersect. Hence

$$P'R' . SQ = Q'S' . PR.$$

In order to obtain the complete theorem for the quadrilateral we have to combine the two theorems enunciated above. Thus, applying the first of the two theorems to the triangles $DP'C$, QRS, we see that if $P'D'C$ be described similar to RQS, then DD', RS have the same product and are equally inclined to the same line as $P'R'$, SQ. Thus, in all, we may construct six lines AA', BB', CC', DD', $P'R'$, $Q'S'$ corresponding to SP, PQ, QR, RS, SQ, PR respectively. Any pair, such as AA', SP, have a fixed product, and are equally inclined to a fixed line. Thus the mutual ratios and inclinations of the six lines AA', BB', etc. are independent of the shape of the quadrilateral $ABCD$ from which they are derived.

An interesting particular case of the theorem is that if P', R' coincide with the same point X, then Q', S' coincide with the same point Y, and also A, A' coincide, etc. If we put $SP = a$, $PQ = b$, $QR = c$, $RS = d$, $SQ = x$, $PR = y$, we may enunciate this case in the following way: *If there is a point X such that the triangles XAB, XCD are similar to x, a, b and x, c, d respectively, then there is also a point Y such that the triangles YBC, YDA are similar to y, b, c*

and y, d, a respectively. The general theorem can be simply deduced from this particular case.

For an extension of the theorem to polygons with any number of sides see the *Proceedings of the London Mathematical Society*, vol. xxviii., p. 442.

F. S. MACAULAY.

45. *To simplify expressions of the form*

$$\frac{a^n(x-b)(x-c)}{(a-b)(a-c)}+\frac{b^n(x-c)(x-a)}{(b-c)(b-a)}+\frac{c^n(x-a)(x-b)}{(c-a)(c-b)}.$$

Denote the given expression by $f(x)$.

Then $f(x)-x^n$ vanishes for the *three* values a, b, c of x. It must also be kept in mind that $f(x)$ is a quadratic function of x.

(i.) Let $n=0$, 1 or 2, then $f(x)-x^n$ must vanish *identically*, and therefore $f(x)=x^n$.

(ii.) Let $n=3$, then $f(x)-x^3=-(x-a)(x-b)(x-c)$, and therefore $f(x)=(a+b+c)x^2-(bc+ca+ab)x+abc$.

(iii.) Let $n=4$, then $f(x)-x^4=-(x+a+b+c)(x-a)(x-b)(x-c)$ (since the coefficient of $x^3=0$), therefore $f(x)=(a^2+b^2+c^2+bc+ca+ab)x^2-\{(a+b+c)(bc+ca+ab)-abc\}x+(a+b+c)abc$.

(iv.) Let $n>4$, then $f(x)=p_{n-2}x^2-(s_2p_{n-3}-s_3p_{n-4})x+s_3p_{n-3}$, where $s_1=a+b+c$; $s_2=bc+ca+ab$; $s_3=abc$, and p_r=sum of homogeneous products of r dimensions of a, b, c.

For $f(x)-x^n=-(x^{n-3}+p_1x^{n-4}+p_2x^{n-5}\ldots p_{n-3})(x^3-s_1x^2+s_2x-s_3)$, where $p_1, p_2, p_3 \ldots p_{n-3}$ are determined by the $n-3$ equations,

$$\begin{aligned}
&p_1-s_1=0,\\
&p_2-p_1s_1+s_2=0,\\
&p_3-p_2s_1+p_1s_2-s_3=0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&p_{n-3}-p_{n-4}s_1+p_{n-5}s_2-p_{n-6}s_3=0.
\end{aligned}$$

Hence $(1+p_1x+p_2x^2\ldots)(1-s_1x+s_2x^2-s_3x^3)=1$,

and therefore $1+p_1x+p_2x^2\ldots=\dfrac{1}{(1-ax)(1-bx)(1-cx)}$,

and therefore p_r=sum of homogeneous products of r dimensions of a, b, c.

Hence $f(x)=(s_2p_{n-5}-s_3p_{n-4}+s_1p_{n-3})x^2+(s_3p_{n-4}-s_2p_{n-3})x+s_3p_{n-3}$
$=p_{n-2}x^2+(p_{n-1}-s_1p_{n-2})x+s_3p_{n-3}$.

Similarly, if $\phi(x)$ denote

$$\frac{a^n(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)}+\frac{b^n(x-c)(x-d)(x-a)}{(b-c)(b-d)(b-a)}+\frac{c^n(x-d)(x-a)(x-b)}{(c-d)(c-a)(c-b)}+\frac{d^n(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)};$$

(i.) if $n=0$, 1, 2, or 3, $\phi(x)=x^n$;

(ii.) if $n=4$, $\phi(x)=(a+b+c+d)x^3-(ab+ac+ad+bc+bd+ad)x^2+(bcd+cda+dab+abc)x-abcd$
$=s_1x^3-s_2x^2+s_3x-s_4$;

(iii.) if $n=5$, $\phi(x)=p_2x^3-(s_1s_2-s_3)x^2+(s_1s_3-s_4)x-s_1s_4$;

(iv.) if $n>5$, $\phi(x)=p_{n-3}x^3+(p_{n-2}-s_1p_{n-3})x^2+(s_3p_{n-4}-s_4p_{n-5})x-s_4p_{n-4}$, where p_r=sum of homogeneous products of r dimensions of a, b, c, d.

E. M. LANGLEY.

46. *Method of obtaining the integral solutions of the indeterminate equation*

$$y^2=ax+b.$$

If the squares of the natural numbers in succession be divided by any number n, the remainders will recur regularly.

For any number $=rn+s$, where $s<n$; and $(rn+s)^2/n$ clearly has the same remainder as s^2/n.

Now, whenever the remainder of $y^2\div a$ is equal to b, x is an integer.

Thus, if $y=s$ gives a remainder b, we may write $y=ar+s$, and we obtain

$$x=ar^2+2rs+(s^2-b)/a.$$

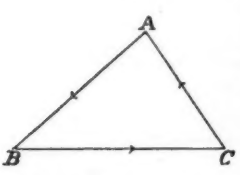
Taking, for example, $y^2=7x+4$, we see that $y=2$, and $y=5$ give the remainder 4.

$$\therefore\quad y=7r+2,\quad \text{or}\quad 7r+5,$$

so that $$y=7r\pm2\quad\text{and}\quad x=7r^2\pm4r.$$

The equation $cy^2=ax+b$ can be reduced to the case above by putting z for cy, so that $$z^2=acx+bc.$$ W. H. Besant.

47. *Theorem of Moments for three forces in equilibrium about any point in their plane.*



Let AB, BC, CA represent the three forces, AB and CA being the lines of action of two of them, O the point in the plane. Then if BC were the line of action of the third force, the algebraic sum of their moments about O would clearly be twice the area of the triangle ABC.

Move BC parallel to itself till it passes through A. Its moment is, if positive diminished, if negative increased, by twice the area of the triangle ABC. Thus when the forces meet in a point the algebraic sum of their moments is zero.

H. A. Roberts.

48. *On the expression of* $\sin\theta$ *as an infinite product.*

Take the expression

$$\left(1-\frac{\sin^2\theta/n}{\sin^2\pi/n}\right)\left(1-\frac{\sin^2\theta/n}{\sin^2 2\pi/n}\right)\ldots \text{ to } \tfrac{1}{2}(n-1) \text{ factors} \quad (n \text{ odd}).$$

Its value is known to be $\sin\theta\div n\sin\theta/n$.

Let r be a positive integer $>\theta/\pi$;

" s " " " $>r$, but $>\frac{1}{2}(n-1)$.

Thus n must be taken $>1+2\theta/\pi$, and θ/n is acute.

The s^{th} factor <1 and $>1-\sin^2\frac{\theta}{n}\operatorname{cosec}\frac{s-1}{n}\pi\operatorname{cosec}\frac{s}{n}\pi$,

or $$>1-\sin^2\frac{\theta}{n}\operatorname{cosec}\frac{\pi}{n}\left(\cot\frac{s-1}{n}\pi-\cot\frac{s}{n}\pi\right),$$

which $$>\left(1-\sin^2\frac{\theta}{n}\operatorname{cosec}\frac{\pi}{n}\cot\frac{s-1}{n}\pi\right)\div\left(1-\sin^2\frac{\theta}{n}\operatorname{cosec}\frac{\pi}{n}\cot\frac{s\pi}{n}\right),$$

provided $\sin^2\frac{\theta}{n}\operatorname{cosec}\frac{\pi}{n}\cot\frac{s-1}{n}\pi<1$.

For if a and b are both positive and <1, $1-a>(1-a-b)/(1-b)$.

Hence, if r is such that $\frac{r\pi}{n}$ (and therefore $\tan\frac{r\pi}{n}$) $>\sin^2\frac{\theta}{n}\operatorname{cosec}\frac{\pi}{n}$, we have that the product of all the factors after the $r^{\text{th}}<1$ and

$$>\left(1-\sin^2\frac{\theta}{n}\operatorname{cosec}\frac{\pi}{n}\cot\frac{r\pi}{n}\right)\div\left(1-\sin^2\frac{\theta}{n}\operatorname{cosec}\frac{\pi}{n}\cot\frac{n-1}{2}\frac{\pi}{n}\right).$$

(The inferior limit assigned to r is $\frac{n}{\pi}.\sin^2\frac{\theta}{n}.\cot\frac{\pi}{n}.\sec\frac{\pi}{n}$, which, since $n=3$ at least, is $<\frac{n}{\pi}.\frac{\theta^2}{n^2}.\frac{n}{\pi}.2$ or $2\theta^2/\pi^2$, a quantity independent of n.)

Thus the product of the first r factors is between

$$\sin\theta\div n\sin\frac{\theta}{n} \text{ and } \sin\theta\div n\sin\frac{\theta}{n}\left(1-\sin^2\frac{\theta}{n}\operatorname{cosec}\frac{\pi}{n}\cot\frac{r\pi}{n}\right).$$

Increasing n indefinitely, we find that

$$\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\ldots\left(1-\frac{\theta^2}{r^2\pi^2}\right)$$

is between $\sin\theta/\theta$ and $\sin\theta/\theta(1-\theta^2/r\pi^2)$, if $r>\theta^2/\pi^2$.

Thus the infinite product $(1-\theta^2/\pi^2)(1-\theta^2/2^2\pi^2)\ldots$ converges to the value $\sin\theta/\theta$.

A like method may be applied to the series

$$\frac{1}{n}.\frac{\sin 2\theta/n}{\sin^2\theta/n-\sin^2\pi/n}+\frac{1}{n}.\frac{\sin 2\theta/n}{\sin^2\theta/n-\sin^2 2\pi/n}+\ldots \text{ to } \tfrac{1}{2}(n-1) \text{ terms},$$

whose value is $\cot\theta-\frac{1}{n}.\cot\theta/n$.

Here we have, if $s>r>\theta/\pi$,

$$\frac{s^{\text{th}}\text{ term}}{r^{\text{th}}\text{ term}}=\frac{\sin^2 r\pi/n-\sin^2\theta/n}{\sin^2 s\pi/n-\sin^2\theta/n}<\frac{\sin^2 r\pi/n}{\sin^2 s\pi/n}$$

$$<\sin^2\frac{r\pi}{n}\operatorname{cosec}\frac{s-1}{n}\pi\operatorname{cosec}\frac{s\pi}{n}$$

$$<\sin^2\frac{r\pi}{n}\operatorname{cosec}\frac{\pi}{n}\left(\cot\frac{s-1}{n}\pi-\cot\frac{s\pi}{n}\right),$$

and so on.

A. C. Dixon.

EXAMINATION QUESTIONS AND PROBLEMS.

Our readers are earnestly asked to help in making this section of the Gazette *attractive by sending either original or selected problems.*

Solutions of problems should be sent within three months of the date of publication. They should be written only on one side of the paper, and without the use of contractions not intended for printing. Figures should be drawn with the greatest care on as small a scale as possible, and on a separate sheet.

The source of selected problems is indicated by—C. (Cambridge), O. (Oxford), D. (Dublin), W. (Woolwich), Sc. (Science and Art Department).

201. The product of the latera recta of the two parabolas through four concyclic points is $\frac{1}{8}(d_1^{\,2}-d_2^{\,2})\sin\omega$, where d_1, d_2 are the diagonals of the quadrilateral formed by the four points, and ω the angle between them. (C.)

202. A quadrilateral $ABCD$ is inscribed to a circle, centre O, and circumscribed to a circle, centre I; OI cuts the circle $ABCD$ in H and K. Show that

$$AI^{-2}+CI^{-2}=HI^{-2}+KI^{-2}. \quad \text{(C.)}$$

203. Find the areal equations of the circles related to the triangle of reference. (C.)

204. If

$$a+b+c+d+e+f+abc+def=0 \text{ and } ab+bc+ca=de+ef+fd$$

then $$(d^2-1)(a+e)(a+f)=(a^2-1)(d+b)(d+c).$$ (C.)

205. Show by means of the rule for finding the H.C.F. of two integral functions $f(x)$ and $F(x)$, which have no common divisor, that other integral functions $\phi(x)$ and $\psi(x)$ may be found such that $f(x)\phi(x)+F(x)\psi(x)=1$ identically. If

$$f(x)=x^3+2x^2+2x+2 \text{ and } F(x)=x^3-2x^2+2x-2,$$

find $\phi(x)$ and $\psi(x)$. Utilize this to find a solution in positive integers of each of the equations

(i.) $187x^2-83y^2=1$, (ii.) $x^2-15521y^2=1$. (C.)

206. The tangent at A to the circumcircle of a triangle ABC meets BC in D, and through D a variable chord DRS is drawn meeting the circle in R and S. If CR, BS meet DA in P, Q respectively, prove that the common chord of the two circles ABC, PQR will pass through a fixed point in AD. (C.)

207. A heavy uniform bar AB of length b is suspended in a horizontal position by two vertical strings each of length b. Determine the couple which must be applied to keep the bar at rest in a horizontal plane after it has been twisted through an angle of 60° round a vertical axis through its centre of gravity. (O.)

208. One of the common chords of a circle and a parabola passes through the centre of the circle, prove that the part of the axis intercepted between the two chords is equal in length to the latus rectum. (O.)

209. If $(\sec\theta+\tan\theta)(\sec\phi+\tan\phi)(\sec\psi+\tan\psi)=1$,

then $$\Sigma\sec^2\theta=1+2\Pi\sec\theta.$$ (O.)

210. If

$$(z+y\cos A)/(y\sin A)=(x+z\cos B)/(z\sin B)$$
$$=(y+x\cos C)/(x\sin C)$$

and $$A+B+C=(2n+1)\pi,$$

then each fraction equals $\Sigma\cot A$. (O.)

211. Construct a triangle having its sides proportional to three integers, and the cosine of an angle equal to a given fraction, positive or negative. C. E. M'VICKER.

212. Solve the equation

$$a_1-\frac{1}{a_2}-\frac{1}{a_3}-\ldots-\frac{1}{a_n}+\frac{1}{x}-\frac{1}{a_n}-\frac{1}{a_{n-1}}-\ldots-\frac{1}{a_1}=0.$$ (C.)

213. The centre of gravity of the triangle ABC is at the centre of a circumscribed ellipse. Show that ρ_A, ρ_B, ρ_C are proportional

to a^3, b^3, c^3 respectively, and that $\rho_A\rho_B\rho_C = R^3$, where ρ_A, etc., are the radii of curvature at A, B, C, and R the circum-radius of the triangle. (C.)

214. A', B', C' are the mid-points of the sides of the triangle ABC. Any transversal through A meets $C'A'$, $A'B'$, in Q, R, respectively; prove that BQ, CR are parallel. G. RICHARDSON.

(No. 145 is a particular case.)

215. If the diagonals of a convex quadrilateral are equal, and perpendiculars be drawn bisecting two opposite sides; then these perpendiculars, measured from their point of intersection, are proportional to the sides on which they stand. F. S. MACAULAY.

(An amended form of 183.)

216. If m edges meet at each vertex of a polyhedron of E edges, and each polygonal face have n sides, then

$$m^{-1}+n^{-1}=E^{-1}+2^{-1}.$$

Hence deduce that only five types of such polyhedra are possible, corresponding to the five regular solids. R. F. MUIRHEAD.

217. Two similar and similarly situated triangles A_1, A_2 are such that the same circle can be described in the one and about the other; compare their areas, and prove that if $A_1 = 4A_2$, the triangles are equilateral. (W.)

(Required analytical and geometrical solutions.)

218. The projections on the four sides of a square of any point on its circum-circle form an orthocentric system.

E. M. LANGLEY.

219. Eliminate θ and ϕ between the equations

$$ax\sec\theta - by\operatorname{cosec}\theta = c^2;\quad ax\sec\phi - by\operatorname{cosec}\phi = c^2;$$

$$\cos\frac{\theta-\phi}{2}\sec\frac{\theta+\phi}{2} = \frac{c}{2};\quad c^2 = a^2+b^2. \qquad \text{(Sc.)}$$

220. The numbers 1, 2, 3, ..., $2n$ are arranged around a circle at equal intervals. Through one of these, h, a diameter is drawn; for what values of n, h will the sum of the numbers on one side be equal to the sum of those on the other. (C.)

SOLUTIONS.

126. *Show how to make a square section of a given regular tetrahedron.* (Edinburgh.)

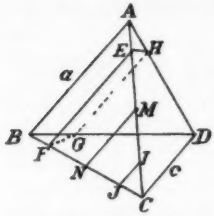
Solution by W. J. GREENSTREET.

Let $EFGH$ be any section parallel to AB and CD. Then EF, GH are parallel to AB; and EH, GF parallel to CD. This is true of any tetrahedron. In a regular tetrahedron AB is perpendicular to CD. In that case $EFGH$ is a rectangle. ACD is equilateral.

Hence $AE = EH = AH = BF$.

Take $CI = CJ = AE$. Then $IJ = EH$.

The sum of the sides of the rectangle is constant, for if MN be the centre-join of AC, BC,



$$BN=NC\,;\ BF=JC\,;$$
$$\therefore\quad EF+IJ=2MN=AB.$$

Hence the rectangle is a maximum when $EF=IJ$, *i.e.* the square section is equidistant from the edges to which it is parallel.

In the general case for any tetrahedron, if $n.AE=m.CE$, then

$$EF.(m+n)=an,\quad EH.(m+n)=mc,$$
$$EF.EH.(m+n)^2=mnac\,;$$

$\therefore$ as $m+n=AC$, the maximum value of the parallelogram is when $m=n$;

and then
$$EF.EH=\frac{ac}{4}.$$

155. For a solution by theory of inversion *v.* Townsend, *Modern Geometry*, vol. i. (1863), p. 205, cor. 4°.

157. *Prove that the six planes perpendicular to the edges of a tetrahedron through their middle points meet in a point.*

Solution by W. J. Greenstreet.

Every point in the plane perpendicular to AB at its middle point is equidistant from A and B, so that the common point of the three planes perpendicular to AB, AC, AD is equidistant from the four vertices A, B, C, D. Hence we see that the six planes meet in a point equidistant from the four vertices, *i.e.* in the centre of the circum-sphere.

160. *If*
$$a+b+c+d+e+f=0,$$
and
$$a^3+b^3+c^3+d^3+e^3+f^3=0,$$
prove that $(a+c)(a+d)(a+e)(a+f)=(b+c)(b+d)(b+e)(b+f)$. (Cambridge.)

Solution by S. O. Roberts, R. Tucker, and G. Heppel.

Let p_r stand for the sum of the products of the four letters, c, d, e, f, taken r at a time. Then $p_1=-(a+b)$.

Also
$$p_1(p_1^2-3p_2)=\Sigma c^3-3p_3=-(a^3+b^3)-3p_3\,;$$
$$\therefore\quad p_3=\tfrac{1}{3}\{(a+b)^3-(a^3+b^3)\}-p_2(a+b)$$
$$=(a+b)(ab-p_2).$$

Hence
$$(a+c)(a+d)(a+e)(a+f)$$
$$=a^4-a^3(a+b)+a^2p_2+a(a+b)(ab-p_2)+p_4$$
$$=a^2b^2-abp_2+p_4$$
$$=(b+c)(b+d)(b+e)(b+f).$$

161. *Find the* H.C.F. *of* $(x^2-a^2-b^2+c^2)^2+4(ax-bc)^2$ *and* $(x-b)^4-(a-c)^4$. (Cambridge.)

Solution by W. E. Jeffares, R. Tucker, and G. Heppel.

$$(x^2-a^2-b^2+c^2)^2+4(ax-bc)^2$$
$$=(x^2+a^2+b^2+c^2)^2-4(a^2+b^2)(x^2+c^2)+4(ax-bc)^2$$
$$=(x^2+a^2+b^2+c^2)^2-4(bx+ac)^2$$
$$=\{(x+b)^2+(a+c)^2\}\{(x-b)^2+(a-c)^2\}.$$

Hence $(x-b)^2+(a-c)^2$ is the required H.C.F.

162. *Prove that*

$$\sin^3\alpha\sin(\beta-\gamma)+\sin^3\beta\sin(\gamma-\alpha)+\sin^3\gamma\sin(\alpha-\beta)$$
$$=-\sin(\alpha+\beta+\gamma)\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta).$$ (Cambridge.)

Solution by G. HEPPEL, W. J. GREENSTREET, and W. E. JEFFARES.

$$\Sigma\sin^3\alpha\sin(\beta-\gamma)=\tfrac{3}{4}\Sigma\sin\alpha\sin(\beta-\gamma)-\tfrac{1}{4}\Sigma\sin 3\alpha\sin(\beta-\gamma)$$
$$=-\tfrac{1}{8}\Sigma\{\cos(3\alpha-\beta+\gamma)-\cos(3\alpha+\beta-\gamma)\}$$
$$=-\tfrac{1}{8}\Sigma\{\cos(3\alpha-\beta+\gamma)-\cos(3\beta+\gamma-\alpha)\}$$
$$=-\tfrac{1}{4}\Sigma\sin(\alpha+\beta+\gamma)\sin 2(\beta-\alpha)$$
$$=-\sin(\alpha+\beta+\gamma)\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta).$$

163. *Sum to n terms*

$$1^2\cos\alpha-2^2\cos 2\alpha+3^2\cos 3\alpha-\ldots.$$ (Cambridge.)

Solution by G. HEPPEL, H. P. KNAPTON, and W. E. JEFFARES.

Put $\alpha=\pi+\beta$, and let S stand for the sum of the series.

Then $$-S=1^2\cos\beta+2^2\cos 2\beta+\ldots+n^2\cos n\beta,$$

$$-2S\cos\beta=1^2(1+\cos 2\beta)+2^2(\cos\beta+\cos 3\beta)+\ldots+n^2(\cos\overline{n-1}\beta+\cos\overline{n+1}\beta)$$
$$=1+2^2\cos\beta+(1^2+3^2)\cos 2\beta+\ldots+(n-1)^2\cos n\beta+n^2\cos(n+1)\beta$$
$$=1+2(1+1^2)\cos\beta+2(1+2^2)\cos 2\beta+\ldots+2(1+n^2)\cos n\beta$$
$$-(n+1)^2\cos n\beta+n^2\cos(n+1)\beta$$
$$=-2S+(1+2\cos\beta+2\cos 2\beta+\ldots+2\cos n\beta)$$
$$-(n+1)^2\cos n\beta+n^2\cos(n+1)\beta,$$

$$2S=\frac{\cos n\beta-\cos(n+1)\beta}{(1-\cos\beta)^2}-\frac{(n+1)^2\cos n\beta-n^2\cos(n+1)\beta}{1-\cos\beta}.$$

164. *If Pp, Qq are normal chords of a conic at right angles to one another, the quadrilateral PQpq has two sides parallel.* (Cambridge.)

Solution by W. J. GREENSTREET and G. HEPPEL.

The tangents and normals at P, Q, form a rectangle $TPOQ$.

Hence $$\frac{OP\,.\,Op}{OQ\,.\,Oq}=\frac{TQ^2}{TP^2}=\frac{OP^2}{OQ^2};$$

$$\therefore\quad\frac{Op}{OP}=\frac{Oq}{OQ};$$

hence PQ, pq are parallel.

165. *Find the maximum and minimum values of*

$$(x-1)^{\frac{1}{4}}(x-2)^{\frac{2}{3}}(x-3)^{\frac{1}{2}}.$$ (Cambridge.)

Solution by G. HEPPEL.

Put $$u=(x-1)^4(x-2)^2(x-3).$$

Taking logs. and differentiating,

$$u_1=u\left(\frac{4}{x-1}+\frac{2}{x-2}+\frac{1}{x-3}\right).$$

This vanishes when $x=1$, $x=2$, and when

$$\frac{4}{x-1}+\frac{2}{x-2}+\frac{1}{x-3}=0. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Differentiating again, and assuming $u_1=0$,

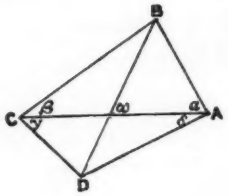
$$u_2=-u\left(\frac{4}{x-1^2}+\frac{2}{x-2^2}+\frac{1}{x-1^2}\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

which is always of opposite sign to u.

One root α of equation (1) lies between 1 and 2, and the other β between 2 and 3; and both of these values make u_2 positive, and u a minimum. Thus u is a maximum when $x=1$ or 2, and a minimum when $x=\alpha$ or β.

166. *Four equal particles, which repel one another with forces varying as the distance, are fastened to an endless string. Prove that in the position of equilibrium the line joining two of the particles is parallel to the line joining the other two.* (Cambridge.)

Solution by G. HEPPEL.



Let T be the excess of tension over repellent force in CD, μAC and μBD the forces in the diagonals.

Then

$$\frac{AB}{AC}=\frac{\sin\beta}{\sin(\alpha+\beta)}; \quad \frac{BD}{AB}=\frac{\sin(\alpha+\delta)}{\sin(\omega-\delta)};$$

$$\frac{\mu AC}{T}=\frac{\sin(\beta+\gamma)}{\sin\beta}; \quad \frac{T}{\mu BD}=\frac{\sin(\omega-\delta)}{\sin(\gamma+\delta)}.$$

Multiplying all together,

$$\sin(\alpha+\delta)\sin(\beta+\gamma)=\sin(\alpha+\beta)\sin(\gamma+\delta),$$

whence $\sin(\alpha-\gamma)\sin(\beta-\delta)=0$, and either $\alpha=\gamma$ or $\beta=\delta$.

168. *The points on a given line, whose distances from two given points have a maximum or minimum ratio, are the ends of a diameter of a circle through the given points.*

Solution by G. HEPPEL, H. P. KNAPTON, and W. E. JEFFARES.

Let A, B be the given points, O that point on the given line such that $OA=OB$, and P any point on the line. Let

$$OA=OB=a,\ AOP=\alpha,\ BOP=\beta,\ OP=x.$$

Then

$$\frac{PA^2}{PB^2}=\frac{a^2+x^2-2ax\cos\alpha}{a^2+x^2-2ax\cos\beta}$$

$$=1-\frac{2a(\cos\alpha-\cos\beta)}{x+a^2/x-2a\cos\beta}.$$

Hence in order that $\frac{PA}{PB}$ may be a $\frac{\text{max.}}{\text{min.}}$, $x+a^2/x$ must be a $\frac{\text{max.}}{\text{min.}}$, and x must be equal to $\pm a$. There are therefore two points P satisfying the given condition, and they lie on the circle with centre O and radius OA or OB, which proves the theorem.

169. *Prove that the equation*

$$\sqrt{a+x}\pm\sqrt{b+x}\pm\sqrt{c+x}\pm\sqrt{d+x}=0$$

has only one root, and find it.

The following solution, which has been communicated by Mr. JEFFARES, is taken from Graham's *Algebra*:

Put

$$\sqrt{a+x}=\beta+\gamma-\alpha, \quad \pm\sqrt{b+x}=\gamma+\alpha-\beta,$$

$$\pm\sqrt{c+x}=\alpha+\beta-\gamma, \quad \pm\sqrt{d+x}=-(\alpha+\beta+\gamma).$$

By squaring and adding, etc., we obtain

$$(a+d)-(b+c)=8\beta\gamma, \text{ etc.}$$

Hence we have $\beta\gamma=a'$, $\gamma a=b'$, $a\beta=c'$,

where a', b', c' are known.

Also we have

$$4x+a+b+c+d=4(a^2+\beta^2+\gamma^2)=4\left(\frac{b'c'}{a'}+\frac{c'a'}{b'}+\frac{a'b'}{c'}\right),$$

giving x, and showing that it has only one value.

174. *If AP, IK be the perpendiculars from the vertex A and in-centre I to the base BC of a triangle ABC, and the tangent to the in-circle parallel to BC meet AP in Q, and the other tangent from Q meet BC in V, prove $VB \cdot VC = VP \cdot VK$.*

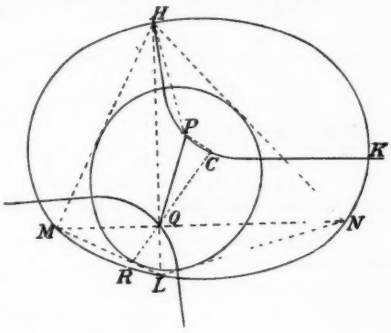
E. BUDDEN.

Solution by PROPOSER.

The tangents intersecting on the fixed line AP determine on BC an involution, centre V. The tangents from P and A intersect BC in P, K and B, C respectively;

$$\therefore \quad VB \cdot VC = VP \cdot VK.$$

176. *If P is a point from which four normals are drawn to a given ellipse, and PQ a diameter of the rectangular hyperbola through the feet of the normals, then, assuming that chords of the ellipse which subtend a right angle at Q envelope another ellipse of which Q is a focus, prove geometrically* (i.) *that P is its second focus,* (ii.) *that CP, CQ are equally inclined to the major axis of the given ellipse,* (iii.) *that CP is to CQ in a constant ratio.* E. P. ROUSE.

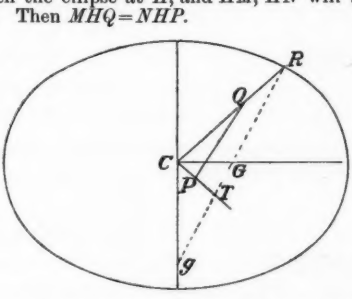


Solution by PROPOSER.

(i.) Let PH, PK be two of the normals from P. Draw chords HQL, MQN at right angles to each other.

PQ is a diameter, therefore HP, HQ are equally inclined to an asymptote, and therefore to the major axis of the ellipse. So also are MN and the tangent at H perpendicular to HP, HQ respectively.

Hence the circle HMN will touch the ellipse at H, and HM, HN will be equally inclined to the major axis. Then $MHQ = NHP$.



Now HM, HN subtending right angles at Q are tangents to the enveloped ellipse, focus Q; therefore the second focus lies on KP, *i.e.* P is the second focus.

(ii.) PQ being a diameter and C a point on the rectangular hyperbola, CP, CQ are equally inclined to an asymptote, and therefore to the major axis of the ellipse.

(iii.) The external diagonal of $HMLN$ is the polar of Q for both in- and circum-ellipses. It is also a directrix of the in-ellipse

because Q is a focus. Hence PQ is perpendicular to it, and therefore parallel to the normal at R, the end of the diameter CQ of the given ellipse. If the normal at R meet CP produced in T, and the axes in G, g,

$$\frac{RG}{Rg}=\frac{b^2}{a^2};\quad \frac{RG}{Gg}=\frac{b^2}{a^2-b^2};\quad \frac{GT}{Tg}=\frac{RG}{Rg}=\frac{b^2}{a^2};\quad \frac{GT}{Gg}=\frac{b^2}{a^2+b^2};\quad \frac{GT}{GR}=\frac{a^2-b^2}{a^2+b^2};$$

$$\therefore\quad \frac{CP}{CQ}=\frac{CT}{CR}=\frac{GT}{GR}=\frac{a^2-b^2}{a^2+b^2}=\text{constant ratio.}$$

If the coordinates of Q are given, we can now find those of P, and the equation to the directrix corresponding to Q, and thence the axes of the enveloped curve. It can also be shown that if the locus of Q is a circle concentric with the given ellipse, the major axis of the enveloped ellipse is constant; while, if the locus of Q is a concentric ellipse similar and similarly situated to the given ellipse, the minor axis is constant.

180. *If the odds against a horse in a race be a to b, then we may call* $\frac{b}{a+b}$ *the apparent chance of that horse winning. Prove that, if the sum of the apparent chances of all the horses in a race be less than unity, one can arrange bets so as to make sure of winning the same sum of money whatever be the issue of the race.* R. W. GENESE.

Solution by F. S. MACAULAY and H. P. KNAPTON.

If the bettor *backs all* the horses, viz., if he bets $\frac{b_r}{a_r+b_r}$ to $\frac{a_r}{a_r+b_r}$ that the r^{th} horse will win; then, if the r^{th} horse wins, the bettor wins $\frac{a_r}{a_r+b_r}$ and loses

$$\Sigma\frac{b}{a+b}-\frac{b_r}{a_r+b_r};$$

$\therefore$ his net winnings are

$$\frac{a_r}{a_r+b_r}-\Sigma\frac{b}{a+b}+\frac{b_r}{a_r+b_r}=1-\Sigma\frac{b}{a+b},$$

which is positive and the same whatever horse wins.

181. *If* Σx^2 *stand for* $x_1^2+x_2^2+\ldots+x_n^2$,
and Σyz *for* $y_1z_1+y_2z_2+\ldots+y_nz_n$,
all the quantities being real, and if

$$\begin{vmatrix}\Sigma x^2, & \Sigma xy, & \Sigma xz\\ \Sigma yx, & \Sigma y^2, & \Sigma yz\\ \Sigma zx, & \Sigma zy, & \Sigma z^2\end{vmatrix}=0,$$

prove that

$$\begin{Vmatrix}x_1, & x_2, \ldots & x_n\\ y_1, & y_2, \ldots & y_n\\ z_1, & z_2, \ldots & z_n\end{Vmatrix}=0.$$

Solution by F. S. MACAULAY.

From the given equation we know that there are three quantities X, Y, Z, satisfying the equations

$$X\Sigma x^2+Y\Sigma xy+Z\Sigma xz=0,$$
$$X\Sigma yx+Y\Sigma y^2+Z\Sigma yz=0,$$
$$X\Sigma zx+Y\Sigma zy+Z\Sigma z^2=0.$$

Multiply these by X, Y, Z respectively, and add. Then

$$X^2\Sigma x^2+Y^2\Sigma y^2+Z^2\Sigma z^2+2YZ\Sigma yz+2ZX\Sigma zx+2XY\Sigma xy=0,$$

i.e.

$$\Sigma(Xx+Yy+Zz)^2=0.$$

Hence we have the n linear equations,

$$Xx_r + Yy_r + Zz_r = 0, \quad (r = 1, 2, 3, \ldots n).$$

These give, by the elimination of X, Y, Z,

$$\begin{Vmatrix} x_1, & x_2, & \ldots & x_n \\ y_1, & y_2, & \ldots & y_n \\ z_1, & z_2, & \ldots & z_n \end{Vmatrix} = 0.$$

Professor A. C. Dixon writes: "The given determinant is the sum of the squares of the determinants of the matrix; therefore," etc.

183. The wording of this question is insufficient. For a correct wording, see No. 215, p. 133.

184. *Show how to inscribe a square in a given quadrilateral.* (D.

Solution by Prof. A. C. Dixon and others.

Let $ABCD$ be the given quadrilateral, supposed convex for simplicity. Take any square $efgh$. On ef, on the same side as the square is, describe a triangle lef, having the angles lef, lfe equal to DBC, DBA respectively. Also on gh, on the same side as the square is, describe a triangle kgh, having the angles kgh, khg equal to BDA, BDC respectively. Let kl cut the circle round lef in b, and the circle round kgh in d; also let be, dh meet in a, and bf, dg meet in c.

Then $abcd$ is similar to $ABCD$. For angle $abd = efd = ABD$, and similarly $adb = ADB$, $cbd = CBD$, $cdb = CDB$. Hence, if on the sides of $ABCD$ we take points E, F, G, H dividing them in the same ratios as the sides of $abcd$ are divided by e, f, g, h, then $EFGH$ will be a square inscribed in $ABCD$.

A similar method is applicable if it is required to inscribe in $ABCD$ a quadrilateral similar to any given quadrilateral. The above construction is one of the simplest; but it has the disadvantage of being indirect.

A direct construction, given by Prof. Dixon, depends on the properties of the centre of similitude. Suppose it required to inscribe in $ABCD$ a quadrilateral similar to $KLMN$. Let P, Q, R be any set of three points on AB, BC, CD respectively, such that PQR is similar to KLM. Then, by a known property, all the triangles PQR have a fixed centre O of similitude, which can be constructed. Also, when O has been found, any number of the triangles PQR, all similar to KLM, can be constructed; and the locus of the point S, such that $PQRS$ is similar to $KLMN$, is a straight line. Find then two of the points S, and join them. This gives the locus of S, and the point where it cuts DA will be a vertex of the required quadrilateral.

The following solution is due to Mr. J. B. Casey. The reader can easily draw the figure.

The sides AB, DC meet in E; AD, BC meet in F. In the triangle AED find O such that

$$AOD = AED + 90°, \quad DOE = FAE + 90°.$$

In the triangle AFB find O' such that

$$AO'F = ABF + 90°, \quad AO'B = AFB + 90°.$$

The circles AOO', OGD, $O'HB$ cut respectively AF, AE in G, H; DE in J; BF in I. Then

$$AOD - AED = 90° = EAO + ODE = HGO + OGJ = HGJ.$$

Similarly $GJH = 45°$, $\therefore GH = GJ$.

In the same way we find

$$GJ = JI \text{ and } GJI = 90°, \quad \therefore GJIH \text{ is a square.}$$

185. *If $a^2, b^2, c^2, x^2, y^2, z^2$ are different from one another and from zero, and if the ratio of*

$$\left(\frac{y}{z}+\frac{z}{y}\right)\left(\frac{z}{x}+\frac{x}{z}\right)\left(\frac{x}{y}+\frac{y}{x}\right) \text{ to } \left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right)$$

is unaltered when x and a are interchanged, prove that

$$\frac{x^2+y^2+z^2+a^2+b^2+c^2}{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}=\frac{b^2c^2y^2z^2}{a^2x^2}.$$

Solution by W. E. JEFFARES.

The ratio of $\Pi\left(\frac{y}{z}+\frac{z}{y}\right)$ to $\Pi\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{\Pi(y^2+z^2)}{\Pi(b^2+c^2)}\cdot\frac{a^2b^2c^2}{x^2y^2z^2}$,

therefore by the conditions of the question

$$\frac{\Pi(y^2+z^2)}{\Pi(b^2+c^2)}\cdot\frac{a^2b^2c^2}{x^2y^2z^2}=\frac{(y^2+z^2)(z^2+a^2)(a^2+y^2)}{(b^2+c^2)(c^2+x^2)(x^2+b^2)}\cdot\frac{b^2c^2x^2}{y^2z^2a^2}.$$

Simplifying, dividing throughout by $(x^2-a^2)a^2x^2$, we get the relation required.

186. *If $ABCD$ is a parallelogram, and EF, GH are any two lines parallel to AB, BC respectively, meeting AB in H, BC in F, CD in G, and DA in E; then AC, EG, HF are concurrent.*

Solution by W. J. GREENSTREET.

Let EG, HF cut AC in O, O'.

By transversals EO, HO' in triangles ADC, ABC,

$$AE.DG.OC=DE.CG.OA\,;\quad BF.HA.O'C=CF.HB.O'A\,;$$

$$\therefore\quad OC/OC'=OA/OA',$$

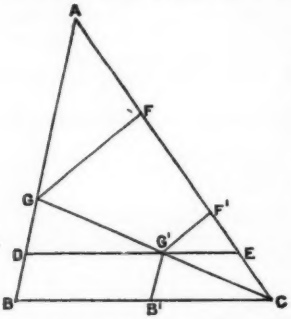
i.e. EG, AC, HF are concurrent at O.

187. *The difference of the sides, including a known angle, of a plane triangle being given, and also the sum of one of those sides and that opposite the given angle, to construct the triangle.*

Solution communicated by Dr. J. S. MACKAY.

In 1774 the question is thus answered by John Turner.

"*Analysis.* Suppose the thing done, and that AGF is the triangle of which are given the angle A, the sum $AF+FG$ and the difference $AG-AF$. Then also will be given $AG+GF$, it being evidently equal to $AF+FG+$the given difference $AG-AF$; and therefore if AB and AC be made respectively equal to the given sums $AG+GF$ and $AF+FG$, the triangle ABC will also be given; and then between the sides of this given triangle we have only to apply GF so as to cut off BG and CF each equal to it; whence appears this easy



"*Construction.* Form the given angle BAC and make BA and AC equal to the two given sums of each side and the base; draw any line DE parallel to BC, and make CF' and $F'G'$ each equal to BD; then draw $CG'G$ and lastly GF parallel to $G'F'$ cutting off the triangle AGF required."

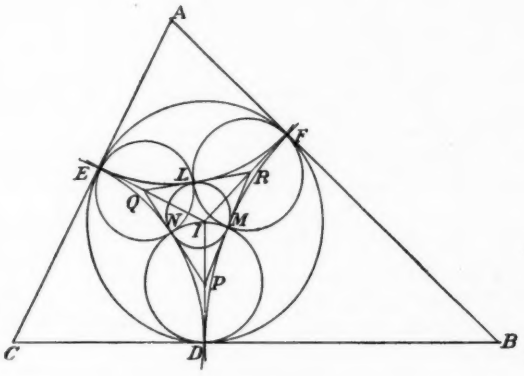
Solution by Trigonometry received from Miss H. A. MOYLAN.

188. *In a given triangle to inscribe three circles touching each other and each touching one side of the triangle at its point of contact with the in-circle.*

E. M. LANGLEY.

Solution by E. P. BARRETT.

Describe circles with vertices A, B, C as centres, AF, BD, CE as radii, where D, E, F are points of in-contact.



Describe a circle to touch these three (M'Dowell, p. 199) at L, M, N.

Draw tangents to this circle at L, M, N. These intersect on ID, IE, IF at P, Q, R. The circles, centre P radius PD, etc., are the circles required.

Professor A. C. Dixon remarks: "Invert from a point of contact of in-circle and side, and the problem becomes—Draw two equal circles touching one another, and touching a given straight line at given points."

190. *Given the trace of any curve $x=f(y)$, and its tangents, show how to construct geometrically the points and tangents of the curve $ax=yf(y)$, a being any constant.*

A. LODGE.

Solution by the PROPOSER.

Let P be any point on the given curve, PM perpendicular to $y=a$, PN perpendicular to axis of y. OM cuts PN in Q. Q is a point on the required curve.

Also, if the tangent at P meet MN in T, TQ will be a tangent at Q to the required curve.

NOTE.—A_p and A_q being the areas of the two curves (P), (Q), the height $\bar{y}$ of the C.G. of (P) is given by $\bar{y} \,.\, A_p = a \,.\, A_q$; and if $\overline{y'}$ be the height of the C.G. of (Q), the radius of gyration (K) of (P) is given by $\dfrac{K^2}{\overline{y'} \,.\, a} = \dfrac{\bar{y}}{a}$. Whence $\bar{y} \,.\, \overline{y'} = K^2$; and if (R), a third curve $x = \dfrac{y^2}{a^2} f(y)$, be similarly derived from Q,

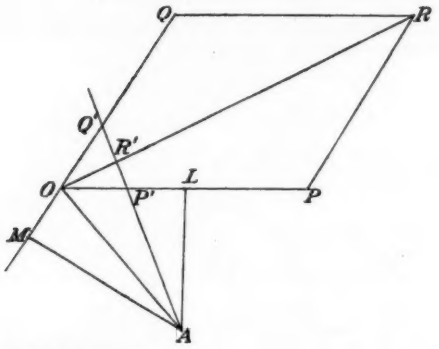
$$K^2(A_p) = a^2(A_r).$$

191. *If OP and OQ be two forces acting at the point O, OR their resultant, and m and n their moments about a point A lying in their plane, prove that one of the anharmonic ratios of the pencil $O(APRQ)$ is equal to m/n.*

J. BRILL.

Solution by PROPOSER.

We have $m = OP \cdot AL$, $n = OQ \cdot AM$.



Further, by Leibnitz' Theorem,

$$\frac{R'Q}{PR} = \frac{OP/OP'}{OQ/OQ'} = \frac{OP \cdot OQ}{OQ \cdot OP'} = \frac{m}{n} \cdot \frac{AM \cdot OQ}{AL \cdot OP'} = \frac{m}{n} \cdot \frac{\Delta AOQ}{\Delta AOP} = \frac{m}{n} \cdot \frac{AQ}{AP'}.$$

Therefore
$$\frac{AP' \cdot R'Q}{AQ' \cdot PR'} = \frac{m}{n}.$$

192. *If the still air at A were transferred to B, β degrees farther north, it would cause a wind of b miles per hour; and if transferred to C, β degrees farther south, it would cause a wind in the contrary direction of c miles per hour; find the latitude of A when the ratio of b to c is 5 to 4 and β is equal to 20.*

W. P. GOUDIE.

Solution by the PROPOSER.

Let ϕ be required latitude, ω the velocity of rotation at equator in miles per hour.

Velocities of point on surface at $\phi, \phi \pm \beta$ respectively are $\omega\cos\phi, \omega\cos(\phi \pm \beta)$.

When air is still

$$b = \omega(\cos\phi - \cos\overline{\phi+\beta}), \quad c = \omega(\cos\overline{\phi-\beta} - \cos\phi), \quad \frac{b}{c} = \frac{\tan\phi + \tan\beta/2}{\tan\phi - \tan\beta/2};$$

$$\therefore \quad \tan\phi = \frac{b+c}{b-c}\tan\frac{\beta}{2} \text{ giving } \phi.$$

Numerical case, $\phi = \tan^{-1}(9\tan 10°)$.

193. *Show, without extracting the roots, that*

$$\sqrt[3]{20 - 14\sqrt{2}} + \sqrt[3]{20 + 14\sqrt{2}} = 4.$$

W. P. GOUDIE.

Solution by W. E. JEFFARES, the PROPOSER, and others.

Let the expression be denoted by x. Cubing,

$$(20 - 14\sqrt{2}) + (20 + 14\sqrt{2}) + 3\{(20 - 14\sqrt{2})(20 + 14\sqrt{2})\}^{\frac{1}{3}}x = x^3,$$

whence $x^3 - 6x - 40 = 0$ or $x = 4$ the real root.

195. This question may be proved in a similar way to No. 158, p. 118. It is also a particular case of the theorem in Note 44, p. 127. Another solution, by C. E. M'VICKER, M.A., has been received.

REVIEWS AND NOTICES.

The Calculus for Engineers. By PROFESSOR JOHN PERRY, M.E., D.Sc., F.R.S. (Edward Arnold.) This excellent work which, as the preface informs us, contains the most important part of the course in the Calculus for Mechanical and Electrical Engineering students at the Finsbury Technical College, should prove a great boon to practical engineers who have not had the advantage of a mathematical training, and to whom the Calculus is more or less of a bugbear. They will rejoice to learn upon such good authority as that of Professor Perry that a knowledge of the differentiation and integration of the three functions x^n, e^{ax}, and $\sin ax$ is sufficient for nearly every practical engineering purpose. The explanation of these operations, and the exemplification of their use in the solution of typical problems in applied mechanics, engineering, and electrical science, form accordingly the greater part of the book.

Mechanical engineers have not, perhaps, generally understood that while a thorough working knowledge of the elementary parts of the Calculus would be of the utmost service to them, there is in general but little need that they should grapple with the higher theorems. "Happy is the engineer who is also a mathematician, but it is given to only a few men to have the two so very different powers."

Professor Perry deprecates the mere attainment of facility in differentiation. It may suit the pure mathematician to deal with abstract expressions and symbols in the light of interesting entities; but the author emphasizes the precept that for the engineer it is of vital importance that at each step he should have a clear mental picture of the practical counterpart of his formulae, and of the operations which his mathematics express.

The constant use of squared paper for running out the curves corresponding to the equations is persistently recommended, and the importance of habitually translating into common language those physical laws which find their complete mathematical expression in differential equations insisted upon. By hard work on these lines the student is urged to acquire a firm grasp of his subject, and to avoid the experience of those who, having proved a theorem, only realize years afterwards precisely what it was they had proved.

In short, the practical engineer will find in this work what he should know of the mathematics of his profession, divested of the superfluous "academic exercise work" which "is mainly used to enable school boys to prepare for examination"; and before working through the whole of the book he will have acquired a useful knowledge of the theory of engineering.

The last chapter is nevertheless devoted to "academic exercises," in the course of which all the fundamental cases of differentiation are given, and a useful list of integrals appended; also integration by parts, partial fractions, maxima and minima, polar coordinates, Taylor's, Maclaurin's, and De Moivre's theorems, elliptic integrals, the Bessel function, and the Gamma function are dealt with.

It is of course easy to suggest the introduction of additional matter into a work of this kind in which the author must have found it difficult to decide how much might safely be omitted. But some important formulae might have been more fully explained. For instance, before the thirty pages or so devoted to beams it would have been of distinct assistance to most students to have had the relation between Young's modulus, the moment of inertia, the radius of curvature, and the bending moment, as expressed in the formulae $\frac{1}{r} = \frac{M}{EI}$, clearly explained and proved.

With regard to a remark, on page 24, that nobody has yet given a name to the rate of change of acceleration, while none appears to have come into general use, we may mention that the term "accelerity" has been used with this meaning; and an ingenious writer has suggested the "hurry."

The remarkable distinctness and simplification of the details of the subject imparted by the author's lucid method and vigorous style are admirably calculated to arouse and keep alive in the student a keen interest in his work. A. L. J.

The Tutorial Trigonometry. By WILLIAM BRIGGS, M.A., F.R.A.S., and G. H. BRYAN, Sc.D., F.R.S. (W. B. Clive.) This Trigonometry is produced in the same style as other volumes in the Tutorial Series. The text is printed in

many different degrees of size and thickness of type, with the object of indicating to the reader the precise degree of importance he should attach to each separate piece of bookwork. This results in the general appearance of the pages being far from attractive. We agree with the authors that the definitions of the trigonometric functions for all angles should be introduced early; but unfortunately their explanation of positive and negative is obscure and incorrect. The articles treating of the ratio of the circumference of a circle to the diameter, though leaving much to be desired, are a step in the right direction. We find, however, the old pretence of proving, what is in reality an axiom, that $\sin\theta/\theta = 1$ when θ is infinitely small. Some of the articles are written with exceptional clearness, notably that on the ambiguous case in the solution of triangles. F. S. M.

Euclid: Books I.-IV. By RUPERT DEAKIN, M.A. (W. B. Clive.) This is a beautifully printed and attractive looking Euclid. Its special feature is an excellent commentary upon and analysis of the propositions, which should prove of great assistance to pupils in obtaining a thorough mastery of the Elements. The additional propositions should have been more numerous and more elementary; and we think it would have been wiser to have left untouched such subjects as continuity, inversion, and reciprocation. F. S. M.

BOOKS, MAGAZINES, ETC., RECEIVED.

First Stage Sound, Light, and Heat, By J. DON, M.A., B.Sc. (W. B. Clive.)

Proceedings of the Edinburgh Mathematical Society. Vol. xv.

Isogonic Centres of a Triangle. By J. S. MACKAY, M.A., LL.D. ("Proceedings E.M.S.," vol. xv.)

The General Wallace Line of an Inscribed Polygon. By DR. N. QUINT. ("Nieuw Archief voor Wiskunde.")

The Foundations of Geometry. A review containing letters from Gauss to W. Bolyai, by Dr. G. B. HALSTED. ("Science," Sept. 24.)

Sul miglior modo di trattare in iscuola la teoria dell'equivalenza. By PROFESSOR RODOLFO BETTAZZI.

Sulla scomposizione dei punti singolari delle superficie algebriche, intorno ad una mia Memoria, "Annali di Matematica," vol. xxv. By PROF. C. SEGRE. ("Atti della R. Acc. delle Sc. di Torino," vol. xxxii.)

Sul genere lineare di una superficie e sulla classificazione a cui esso dà luogo. By PROF. G. CASTELNUOVO. ("Rendiconti della R. Acc. dei Lincei," vol. vi., serie 5ª.)

Alcune proprietà fondamentali dei sistemi lineari di curve tracciati sopra una superficie algebrica. By PROF. G. CASTELNUOVO. "Annali di Matematica," serie 2, vol. xxv.)

The American Mathematical Monthly. May to July, 1897. Edited by PROF. B. F. FINKEL and J. M. COLAW, A.M. (A. Hermann, 8 Rue de la Sorbonne, Paris.)

Journal de Mathématiques Élémentaires. June to September, 1897. Edited by PROF. M. DE LONGCHAMPS. (Librairie Ch. Delagrave, 15 Rue Soufflot, Paris.)

Periodico di Matematica. July to October, 1897. Edited by PROF. GIULIO LAZZERI. (Tipografia di Raffaello Giusti, Livorno.)

Bolletino della Associazione Mathesis. No. 4.

The following *Mathematical Notes* have been received:

On a class of algebraic functions. By PROF. H. W. LLOYD TANNER, M.A.

On the Simson line, etc. By W. J. GREENSTREET, M.A.

Proof of a formula for the volume of a tetrahedron.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO.